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SUPERCONDUCTIVITY AND ELECTRIC FIELDS: A RELATIVISTIC EXTENSION OF BCS SUPERCONDUCTIVITY

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The effects of static electric fields on the superconducting state are studied within a relativistic extension of the BCS theory of superconductivity.

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1. Introduction

The Bardeen–Cooper–Schrieffer (BCS) theory[1] of 1957 provides a microscopic understanding for the phenomena of Low Temperature (LTc) superconductivity.[2,3] Below the critical temperature T_c, attractive electron-phonon interactions lead to electron-electron Cooper pair formation in the s-wave channel. The ensuing Cooper pair condensation of such identical bosonic quantum states implies the appearance of an energy gap, associated to a spontaneous breaking of the U(1) local gauge symmety of the electromagnetic interaction,[4] hence also an effective non-zero mass for the photon which translates into the physical Meissner effect[5] of magnetic field screening in any bulk superconductor. The existence of a gap $\Delta(\vec{r})$ also ensures the phenomenon of perfect conductivity, through the collective dynamics of the condensed Cooper pair electrons for electric currents less than some critical value.

The gap $\Delta(\vec{r})$ may also be given the interpretation, up to normalisation, of the common complex valued quantum wave function of the spin 0 Cooper pairs. It also plays the role of an order parameter for the phase transition towards the superconducting state, of relevance in an effective field theory description. Among the successes of the BCS theory in the weak coupling regime, one finds the correct description of the temperature dependence of the order parameter, hence the identification of the critical temperature, the critical magnetic field for the Meissner effect, and consequently also the critical current, inclusive of subtle effects such as the isotopic dependence of the critical temperature.

In effect, the superconducting state is understood in terms of a coherent superposition of electron-electron pairs of which the momentum and spin values are coupled in order to build up a spin 0 state of vanishing total momentum, in the absence of any electric current,

$$\prod_{\vec{k}} \left[u(\vec{k}) + e^{i\theta(\vec{k})} v(\vec{k}) \, c_\downarrow^\dagger(-\vec{k}) c_\uparrow^\dagger(\vec{k}) \right], \tag{1}$$

where $c_\uparrow^\dagger(\vec{k})$ and $c_\downarrow^\dagger(\vec{k})$ represent the creation operators of electron states of momentum $\vec{k}$ and spin projection up or down, respectively, while $u(\vec{k})$ and $v(\vec{k})$ stand for the probability amplitudes of occupation of states without or with a single Cooper pair of vanishing total momentum and spin. These two functions are identified through a gap equation expressing the minimisation of the energy of such a trial state with respect to these two functions obeying a normalisation condition involving the combination $|u(\vec{k})|^2 + |v(\vec{k})|^2$.

A few years after the formulation of the BCS theory, Gor'kov showed[6] how through a finite temperature quantum field theory approach, it is possible to construct an effective field theory representation of the microscopic dynamics, which for all practical purposes coincides with the famous Ginzburg–Landau (GL) phenomenological description of superconductivity dating back to 1950 already.[7] Based on Landau's approach towards a general theory of phase transitions, within the GL theory the free energy density of the superconducting state, F_s, compared to that of the normal state, F_n, is expressed as a functional of the order parameter $\psi(\vec{r})$ which, up to normalisation, is identified with the superconducting gap $\Delta(\vec{r})$,

$$F_s - F_n = \alpha|\psi|^2 + \frac{1}{2}\beta|\psi|^4 + \frac{\hbar^2}{2m}\left|\left(\vec{\nabla} - i\frac{q}{\hbar}\vec{A}\right)\psi\right|^2 + \frac{1}{2\mu_0}\left(\vec{B} - \vec{B}_{\text{ext}}\right)^2 , \quad (2)$$

where α and β are temperature dependent coefficients defining an effective potential energy density

$$V\left(|\psi|\right) = \alpha|\psi|^2 + \frac{1}{2}\beta|\psi|^4 , \quad (3)$$

while the notation for the other quantities is standard, and corresponds to the magnetic vector potential, $\vec{A}$, the magnetic induction, $\vec{B}$, and the externally applied magnetic induction, $\vec{B}_{\text{ext}}$, with μ_0 being the vacuum magnetic permittivity. Finally, $q = -2|e|$ and m stand, respectively, for the Cooper electric charge and effective mass in the conducting material. Given the above potential energy, as soon as the parameter α turns negative below a specific critical temperature T_c, $\alpha(T < T_c) < 0$, one has a potential of the Higgs type with minima attained for nonvanishing expectation values of ψ, thereby spontaneously breaking the local U(1) phase invariance symmetry of the GL functional.

As explained in any standard textbook on superconductivity,[2,3] the phenomena of perfect conductivity and diamagnetism are readily established from the GL equations for the order parameter, namely the variational equations of motion stemming from the GL functional,

$$-\frac{\hbar^2}{2m}\left(\vec{\nabla} - i\frac{q}{\hbar}\vec{A}\right)^2\psi(\vec{r}) + \alpha\,\psi(\vec{r}) + \frac{1}{2}\beta\,|\psi(\vec{r})|^2\,\psi(\vec{r}) = 0,$$

$$\vec{J}(\vec{r}) = \frac{1}{\mu_0}\vec{\nabla}\times\vec{B}(\vec{r}) \quad (4)$$

$$= -i\frac{q\hbar}{2m}\left(\psi^*(\vec{r})\vec{\nabla}\psi(\vec{r}) - \vec{\nabla}\psi^*(\vec{r})\,\psi(\vec{r})\right) - \frac{q^2}{m}\,|\psi(\vec{r})|^2\,\vec{A}(\vec{r}),$$

with boundary conditions requiring that the current $\vec{J}(\vec{r})$ has a vanishing component normal to any boundary in the case of a finite domain. In par-

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ticular, space dependence of the order parameter may then be accounted for, so that not only is the Meissner effect characterized by the magnetic penetration length λ, but coherence effects are also characterized by a coherence length ξ, with their ratio distinguishing between Type I and Type II superconductors. Namely, given the GL parameter $\kappa = \lambda/\xi$, Type I superconductors correspond to a value of the GL parameter less than $1/\sqrt{2}$, $\kappa < \kappa_c = 1/\sqrt{2}$, and Type II superconductors to a value larger than κ_c, $\kappa > \kappa_c$. The manner in which magnetic fields may or may not penetrate such materials in their bulk is different for each Type. In particular, Type II materials sustain Abrikosov vortices,[8] namely flux tubes carrying a unit value of the quantum of flux penetrating the material even in the superconducting state.

Furthermore, in the limit that any spatial dependence of the order parameter may be ignored, the GL equations lead back to yet an older empirical approach to superconductivity from 1935 due to the London brothers.[9] The London equations simply read

$$\vec{E} = \frac{\partial}{\partial t}\left(\Lambda \vec{J}\right), \qquad \vec{B} = -\vec{\nabla} \times \left(\Lambda \vec{J}\right), \tag{5}$$

$\vec{E}$ and $\vec{B}$ being the electric and magnetic fields, respectively, $\vec{J}$ the current density, and Λ a phenomenological parameter given by

$$\Lambda = \frac{m}{n_s q^2}, \tag{6}$$

n_s being the density of superconducting electrons. Perfect conductivity is a direct consequence of the first London equation, while the Meissner effect follows from the second with a magnetic penetration length λ_L such that

$$\lambda_L^2 = \frac{m}{\mu_0 n_s q^2}. \tag{7}$$

Of course, it is to be understood that all the above descriptions and their ensuing equations of motion are also coupled to Maxwell's equations of electromagnetism,

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0}\,\rho, \quad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = \vec{0},$$

$$\vec{\nabla} \cdot \vec{B} = 0, \quad \vec{\nabla} \times \vec{B} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J}, \tag{8}$$

ρ being the electric charge density, and ϵ_0 et μ_0 the usual electric and magnetic permittivity properties of the vacuum such that $\epsilon_0 \mu_0 = c^2$, c being the speed of light in vacuum.

But one of the players in these latter equations which is conspiscuously missing from the above discussion of available descriptions of LTc superconductivity is the electric field. How do electric fields influence or affect the electromagnetic properties of superconducting materials? The usual answer[2,3] to this question is that, in the stationary state without any time dependence, electric fields cannot have any effect whatsoever since, according to the first London equation and because of the perfect conductivity of any superconductor, the electric field must vanish identically at least for bulk materials. Given that this answer is presumably acceptable, it then remains nonetheless possible that for nanoscopic materials of increasing use and interest in nanotechnology, electric fields may have some effect close to the surface of such materials, since it would be difficult to imagine how an externally applied electric field could abruptly and discontinuously vanish when moving from the outside to the inside of such a conductor.[10] In the present contribution, we briefly discuss this question, and present some of the conclusions that have been reached through the work of which far more details may be found in Ref. 11.

The characterisation of the problem is presented in Sec. 2. Next, a first framework in which to address the issue is briefly considered in Sec. 3, with experimental results proving that the analysis must be extended to include the effects of all electrons of a superconducting material, even the "normal" ones. An appropriate framework is then developed in Sec. 4, leading first to the identification of the effective potential energy in analogy with the GL potential, and next, in Sec. 5, some further dynamical properties of the superconducting state in the presence of an applied electric field. Finally, Sec. 6 offers some conclusions and prospects for further work along similar lines.

2. The Problem

To highlight the issue mentioned above from different points of view, let us first recall that the relativistic covariance properties of Maxwell's equations are best made manifest through the fact that the electromagnetic scalar and vector potentials, Φ and $\vec{A}$, respectively, define the components of a 4-vector as

$$A^\mu = \left(\frac{\Phi}{c}, \vec{A} \right), \qquad \mu = 0, 1, 2, 3, \tag{9}$$

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while the associated field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ (with $x^\mu = (ct, \vec{x})$) is directly related to the electric and magnetic fields $\vec{E}/c$ and $\vec{B}$ as

$$\frac{\vec{E}}{c} = -\vec{\nabla}\frac{\Phi}{c} - \frac{\partial}{\partial(ct)}\vec{A}, \qquad \vec{B} = \vec{\nabla} \times \vec{A}. \tag{10}$$

The fact of the matter is that all the approaches briefly reviewed in the Introduction are intrinsically nonrelativistic, but are nonetheless coupled to the relativistic covariant Maxwell's equations. In itself this is not necessarily problematic provided the considered regime remains nonrelativistic in a sense to be specified. However, since a nonrelativistic limit amounts to taking a limit such that $1/c \to 0$, clearly any of the effects related to the electric scalar potential, Φ/c, and field, $\vec{E}/c$, in the same units as those of the magnetic sector, decouple in such a limit. Electric field effects in superconductors are thus at best subleading in $1/c$, but not necessarily vanishing altogether. Therefore one ought to develop a manifestly relativistic invariant framework in which to analyse such effects.

From yet another point of view, one may also argue for the necessity of such a framework by considering specific experimental set-ups.[10] For example, imagine an infinite slab subjected to an external magnetic field parallel to it, without an external electric field being applied in the laboratory frame. Due to the Meissner effect, the magnetic field will penetrate the slab only up a typical distance set by the magnetic penetration length λ. Imagine now performing a Lorentz boost in a direction both parallel to the slab and perpendicular to the magnetic field. In such a boosted frame, not only is the strength of the magnetic field slightly modified, but more importantly there appears now an electric field perpendicular to the slab, and *a priori* also inside the superconductor, and thus necessarily with precisely the same penetration length as the magnetic field! Hence, if for such a *gedanken experiment* it is justified to restrict only to electrons in the superconducting state, one is forced to conclude, on basis of relativistic covariance, that an electric field does not necessarily vanish inside a superconductor, and does penetrate such materials with a penetration length identical to the magnetic one.

Such considerations thus call for a framework in which Maxwell's electromagnetism is coupled to the superconducting state in a manifestly relativistic invariant manner. Such a framework may also be of relevance to other issues of superconductivity, especially for heavy metallic coumpounds corresponding to chemical elements of large Z values, implying significant relativistic corrections to electronic orbital properties.[12,13]

In effect, an experiment such as the one described above has been performed using a nanoscopic slab of superconducting aluminium, subjected to both a magnetic field parallel to the slab and an electric field perpendicular to it (for the details, see Ref. 11). Much to our surprise, no effect of the electric field whasoever, even when ramped up to considerable values, was observed on the critical temperature for the superconducting state, while the latter's values and dependence on the magnetic field were properly observed, measured and seen to coincide with established values for the critical temperature in the case of that material.

The experiment was analysed within the framework of both the non-relativistic GL approach, and its obvious covariant generalisation through the U(1) Higgs model for a charged complex scalar field with Lagrangian density functional[4,10,14]

$$\mathcal{L} = -\frac{1}{4}\epsilon_0 c F^{\mu\nu} F_{\mu\nu} + \frac{1}{2}\epsilon_0 c \left(\frac{\hbar}{q\lambda}\right)^2 \left\{\left|\left(\partial_\mu + i\frac{q}{\hbar}A_\mu\right)\psi\right|^2 - \frac{1}{2\xi^2}\left(|\psi|^2 - 1\right)^2\right\}.$$
(11)

In the latter case, indeed the London equations are modified in the manner expected on account of manifest Lorentz covariance, with in particular necessarily identical electric and magnetic penetration lengths.[10]

Either framework predicts effects of an electric field on such an experimental set-up, with specific characterisations of these effects as a function of the temperature.[10] And much to our surprise, no effect whasoever was observed in spite of repeated and carefully prepared measurements and nanoscopic samples of the aluminium slab.[11]

Faced with this conundrum, we were led to the necessity of developing a microscopic model of s-wave superconductivity which is manifestly relativistic invariant and accounts for all electronic states, whether "superconducting" or "normal" states, the latter being the main suspects as being behind the close-to-perfect screening of any elecric field however large. In other words, a relativistic extension of the BCS theory for LTc superconductors is *a priori* required to account for the experimental results.

3. A Model

The superconducting state we are interested in being in thermodynamical equilibrium, the natural framework to model the problem at the microscopic level is in terms of Finite Temperature Quantum Field Theory (FTQFT),[15,16] in which electron states are described by the Dirac field coupled to the electromagnetic field in a U(1) gauge invariant manner. One

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needs to compute the partition function of the system, in the presence of stationary background electric and magnetic fields, as well as a chemical potential[16,17] μ for the electron states, namely

$$Z(\beta) = \operatorname{Tr} e^{-\beta(H-\mu N)}, \qquad \beta = \frac{1}{kT}, \tag{12}$$

H being the Hamiltonian of the system, N its electron number operator, T the absolute temperature and k Boltzmann's constant, while the trace is over all quantum states of the system. The calculation of such a partition function proceeds through both operatorial and path integral techniques.

The Hamiltonian to be used stems from the Lagangian density describing the microscopic dynamics

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \overline{\psi} \left(i\gamma^\mu \partial_\mu - m \right) \psi - e A_\mu \overline{\psi} \gamma^\mu \psi + \mathcal{L}_{4f}. \tag{13}$$

Henceforth, natural units such that $\hbar = 1 = c$ and $\epsilon_0 = 1 = \mu_0$ are used. Here, ψ stands for a Dirac 4-spinor for the Dirac-Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, $\eta^{\mu\nu}$ being the four-dimensional Minkowski spacetime metric of mostly negative signature (for the space components), m is the electron mass and $e < 0$ its electric charge, while $\overline{\psi} = \psi^\dagger \gamma^0$. Finally, $\mathcal{L}_{4f}$ stands for the four-fermion effective interaction to be used to model the phonon-mediated electron-electron interaction responsible for the superconducting state.

A priori, the 4-fermion interaction may involve some combination of all possible relativistic invariant 4-electron operators, of the form,

$$\mathcal{L}_{4f} = g_1 \left(\overline{\psi}\psi \right)^2 + g_2 \left(\overline{\psi}\gamma_5\psi \right)^2 + g_3 \left(\overline{\psi}\gamma^\mu\psi \right)^2 + g_4 \left(\overline{\psi}\gamma^\mu\gamma_5\psi \right)^2 + g_5 \left(\overline{\psi}\sigma^{\mu\nu}\gamma_5\psi \right)^2, \tag{14}$$

with, as usual, $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and $\sigma^{\mu\nu} = i[\gamma^\mu, \gamma^\nu]/2$, and g_i ($i = 1, 2, 3, 4, 5$) arbitrary real couplings constants. However, since the model describes a single fermionic species of which the field degrees of freedom are represented by Grassmann odd variables, and applying Fierz identities, it follows that the same effective interaction may be brought into the form

$$\mathcal{L}_{4f} = \beta_1 \left(\overline{\psi_c}\psi \right)^\dagger \left(\overline{\psi_c}\psi \right) + \beta_2 \left(\overline{\psi_c}\gamma_5\psi \right)^\dagger \left(\overline{\psi_c}\gamma_5\psi \right)$$
$$+ \beta_3 \left(\overline{\psi_c}\gamma^\mu\gamma_5\psi \right)^\dagger \left(\overline{\psi_c}\gamma_\mu\gamma_5\psi \right), \tag{15}$$

where $\psi_c = \eta_c C \overline{\psi}^T$, η_c being an arbitrary phase factor and C the charge conjugation matrix operator. A detailed analysis of the particle and spin content of these different operators in a nonrelativistic limit shows that, in the order in which they appear in the above relation, the first accounts for

a *p*-wave order parameter, the second the *s*-wave BCS state, and the third a superposition of *d*- and *s*-wave contributions. This classification does not coincide with different conclusions available in the literature, which failed to account for the Grassmann character of the electron field and misidentified the proper properties of such a classification under parity.[12,13]

Focusing on *s*-wave BCS superconductivity, we are thus led to consider the following 4-fermion effective interaction

$$\mathcal{L}_{4f}^{BCS} = -\frac{1}{2}g \left(\overline{\psi_c}\gamma_5\psi\right)^\dagger \left(\overline{\psi_c}\gamma_5\psi\right), \tag{16}$$

g being a real coupling constant, in order to model the phonon mediated interaction between electron pairs and leading *in fine* to the Cooper pair condensed state below the critical temperature.

Given that choice as well as well established techniques of FTQFT,[16] it follows that the partition function of the system may be given the following path integral representation,

$$Z(\beta) = \int \mathcal{D}[\psi, \overline{\psi}, \Delta, \Delta^\dagger] \, e^{-\int_0^\beta d\tau \int d^3\vec{x}\,\mathcal{L}_E}, \tag{17}$$

where

$$\mathcal{L}_E = \frac{1}{2}\left[\psi^\dagger\partial_\tau\psi - \partial_\tau\psi^\dagger\psi\right] - \frac{1}{2}i\left[\overline{\psi}\vec{\gamma}\cdot\vec{\nabla}\psi - \vec{\nabla}\overline{\psi}\cdot\vec{\gamma}\psi\right]$$

$$+m\overline{\psi}\psi + eA_\mu\overline{\psi}\gamma^\mu\psi - \mu_0\psi^\dagger\psi \tag{18}$$

$$+\frac{1}{2g}|\Delta|^2 - \frac{1}{2}\left[\Delta^\dagger\left(\overline{\psi_c}\gamma_5\psi\right) + \Delta\left(\overline{\psi_c}\gamma_5\psi\right)^\dagger\right],$$

while from now on μ_0 stands for the chemical potential in the absence of external electromagnetic fields. Here, τ is the imaginary time parameter in which bosonic fields must be periodic and fermionic ones antiperiodic with period β, while Δ is an auxiliary field which is introduced in order to express the 4-fermion interaction in terms of only quadratic couplings of the Dirac field. The ensuing Grassmann odd gaussian integrals are then readily feasable, leading to an effective action for the auxiliary field Δ. As a matter of fact, the field Δ coincides thus with the order parameter of the superconducting state, and measures, up to normalisation, the local density of Cooper states in that state. The effective action obtained through the integration over all fermionic degrees of freedom thus corresponds, in the relativistic setting, to the GL action functional in the nonrelativistic setting. In effect, this is also how Gor'kov established the GL effective description from the BCS microscopic one.[6]

4. The Effective Potential

Before addressing the issue of the external field dependence of the effective action, it is of interest to identify the effective potential independently of such external electromagnetic disturbances and spatial variations of the order parameter. For all practical purposes, this effective potential should correspond to the GL potential of the Higgs type in (3).

In such specific circumstances, given the absence of external electromagnetic fields and the assumption of a space independent order parameter Δ_0, the calculation may be performed exactly through operator techniques by relying on a Bogoliubov transformation which enables one to identify the associated Cooper pairs in analogy with (1). Details may be found in Ref. 11. What such a Bogoluibov transformation achieves is an exact diagonalisation of the quantum Hamiltonian under the above circumstances, with the Cooper pair condensate defining the physical ground state. Furthermore, excitations of the Cooper pair condensate correspond to collective modes of the electron system of definite momentum and charge, hence also of definite energy, known as pseudo-particles. Any of these states, whether the Cooper pair ground state or its pseudo-particle excitations, are obtained as coherent superpositions of the modes of the original free quantum electronic states of the Dirac spinor and its perturbative vacuum.

Denoting by V the volume of the superconductor, the effective potential energy density is expressed as

$$\frac{1}{V} S_{\text{eff}}^{(0)} = \frac{1}{2g} |\Delta_0|^2 + \int \frac{d^3 \vec{k}}{(2\pi)^3} \left[2\omega(\vec{k}) - E_B(\vec{k}) - E_D(\vec{k}) \right]$$

$$-2\frac{1}{\beta} \int \frac{d^3 \vec{k}}{(2\pi)^3} \ln \left[1 + e^{-\beta E_B(\vec{k})} \right] - 2\frac{1}{\beta} \int \frac{d^3 \vec{k}}{(2\pi)^3} \ln \left[1 + e^{-\beta E_D(\vec{k})} \right], \tag{19}$$

where

$$E_B(\vec{k}) = \sqrt{\left(\omega(\vec{k}) - \mu_0 \right)^2 + |\Delta_0|^2},$$

$$E_D(\vec{k}) = \sqrt{\left(\omega(\vec{k}) + \mu_0 \right)^2 + |\Delta_0|^2}, \tag{20}$$

$$\omega(\vec{k}) = \sqrt{\vec{k}^2 + m^2}.$$

Here, $E_B(\vec{k})$ and $E_D(\vec{k})$ stand for the energies of the pseudo-particle excitations of electron and positron type, respectively, in presence of the Cooper pair condensate Δ_0 which clearly specifies also the gap value in the disper-

sion relations for such collective excitations of the superconducting state. Note that the chemical potential μ_0, in the case of an electron conductor,[a] is bounded below by m, since the electron's relativistic rest mass energy must be added to the ordinary nonrelativistic chemical potential value in the present Lorentz covariant framework.

By minimisation of the effective potential and applying the usual BCS approximation which consists in restricting the momentum integration to a region surrounding the Fermi level with a cut-off set to coincide with the Debye lattice frequency to which a Debye energy ξ_D is associated,[1-3] one obtains the following gap equation for the order parameter Δ_0,

$$\frac{1}{2} \int_{-\xi_D}^{\xi_D} d\xi \left\{ \frac{1}{\sqrt{\xi^2 + |\Delta_0|^2}} \tanh \frac{1}{2}\beta\sqrt{\xi^2 + |\Delta_0|^2} \right\} = \frac{1}{gN(0)}, \qquad (21)$$

$N(0)$ being the density of states at the Fermi level. In this expression, possible contributions due to positron-like states are not retained since their value is totally insignificant in the case of an ordinary LTc superconductor. This gap equation may be seen to coincide with the usual BCS gap equation.[1] In particular, in the weak coupling regime and at $T = 0$ K, its solution reads

$$|\Delta_0(0)| \simeq \frac{\pi e^{-\gamma}}{\beta_c} \frac{1}{1 - e^{2/gN(0)}}, \qquad (22)$$

γ being the Euler constant, $\gamma \simeq 0.577$, and $\beta_c = 1/kT_c$. Hence, one recovers the BCS results, and the order parameter interaction chosen in (16) does indeed represent s-wave BCS Cooper pairs within the present relativistic framework. Given the kinematical regime in which the model is being considered, relativistic corrections to the effective potential thus prove to be totally insignificant.

Even though we shall refrain by lack of space from presenting here graphs of the effective potential (which would be quite illustrative of the physical results, for which the interested reader is again referred to Ref. 11), the effective potential (20) is indeed of the Higgs type below the critical temperature, namely with an absolute minimum for a nonvanishing order parameter Δ. However, although a quartic approximation of the Higgs form

$$V_{\text{eff}}^{(1)}(x) = F(x_0) + [F(0) - F(x_0)] \left[\frac{x^2}{x_0^2} - 1 \right]^2, \qquad x = |\Delta|, \qquad (23)$$

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[a]In the case of a positron conductor, μ_0 would be bounded above by $-m$.

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is quite satisfactory for temperatures T sufficiently close to T_c and Δ values sufficiently close to the solution to the gap equation (21), and in which the coefficients $F(0)$ and $F(x_0)$ are chosen to coincide with the values of $V_{\text{eff}}^{(1)}(x)$ for $x = 0$ and $x = x_0$, $x_0 = |\Delta_0|$ standing for the solution to the gap equation, better approximations are possible, which greatly extend the temperature range below T_c for which for all practical purposes the approximation coincides with the exact effective potential given by its integral definition in (20). Possible examples are[11]

$$V_{\text{eff}}^{(2)}(x) = F(x_0) + [F(0) - F(x_0)] \left[\frac{\ln(x_0^2 + \lambda x_0^2) - \ln(x^2 + \lambda x_0^2)}{\ln(x_0^2 + \lambda x_0^2) - \ln(\lambda x_0^2)} \right]^2 , \quad (24)$$

$$V_{\text{eff}}^{(3)}(x) = F(x_0) + [F(0) - F(x_0)] \left[1 - \left(\frac{\left[1 + \gamma \frac{x^2}{x_0^2} \right]^{\alpha} - 1}{[1 + \gamma]^{\alpha} - 1} \right)^2 \right] , \quad (25)$$

where λ, α and γ are parameters whose values and temperature dependence may be fitted[11] to the exact effective potential, leading to very efficient approximations to the exact expression, reliable in far greater temperature and order parameter ranges away from their critical values than the usual Higgs potential of the quartic type, $V_{\text{eff}}^{(1)}(x)$. A study of the phenomenological consequences of such generalised Higgs-like potentials could be of interest, in particular for what concerns their vortex solutions.[14,18,19]

5. The Effective Action

By including the effects of external electromagnetic fields, a computation of the full effective action, and not only the effective potential, is feasible through a perturbative expansion. Namely by also including effects due to space gradients both in the order parameter and the electromagnetic potentials Φ and $\vec{A}$, it is possible to obtain explicit expressions for all physically relevant parameters which empirically characterise the superconducting state. Thus not only are the coherence and magnetic penetration length values and their temperature dependences obtained, but also those of the electric penetration length. Furthermore, other characterisations also become accessible, which are usually not discussed in the literature by lack of interest in the possible effects of electric fields on superconductors. For instance, it is also possible to study how the total electron charge density distribution, which ought to balance the background lattice charge distribution, is accounted for by the order parameter ("superconducting electrons"),

and thus how superconductors could locally acquire charge in specific circumstances. Likewise, a study of the dependence of all the above quantities on the chemical potential μ_0 is also feasible, and is of interest since varying its value amounts to depleting the superconductor of its electrons, or else increasing that number, in other words, charging the material.

By lack of space, all the results obtained so far along such lines are not detailed here. They are available in Ref. 11 together with relevant and illustrative graphs. In the case study of aluminium, experimental values for the magnetic penetration length and its temperature dependence are well reproduced from our analysis when proper account is given of the role of impurity electron rescattering. For what concerns the electric penetration length, our analysis reveals that this observable, heretofore never computed in the literature, receives two types of contributions, in contradistinction to the magnetic penetration length of which the value is solely dependent on the Cooper pair condensate density $|\Delta_0|$. Indeed, for the electric penetration length, not only is there a contribution akin to the magnetic penetration length as expected by reason of the Lorentz covariance arguments discussed in Sec. 2, but in addition the ordinary Thomas–Fermi screening effect existing in normal conductors[20] is also at work. Since typically values for the latter screening length are on the order of the Angström or in fact even less, while magnetic penetration lengths typically range in the tens to hundreds of nanometers, and since their combined effect which finally sets the electric penetration length derives essentially from the sum of their squared inverse values, namely

$$\lambda_{\text{electric}}^{-2} = \lambda_{\text{magnetic}}^{-2} + \lambda_{\text{Thomas–Fermi}}^{-2}, \tag{26}$$

it follows that it is the Thomas–Fermi screening effect which by far and large dominates the screening effects of electric fields in superconductors. This conclusion also explains the null results of our experimental measurements mentioned in Sec. 2. In other words, Cooper pair contributions, namely "superconducting electron" contributions are indeed similar in value for both the magnetic and electric penetration lengths, but in the latter case contributions from "normal electrons" are also involved, and their effect being so overwhelming in the case of that observable, in effect the complete superconducting electric penetration length essentially coincides with the Thomas–Fermi screening length of the conductor even in the normal conducting state.

In fact, since all electron states are being integrated out in the path integral leading to the effective action, the distinction between "supercon-

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ducting" and "normal" electrons is a matter of convention and arbitrary definition, possible for instance by comparing local charge distributions to the background lattice charge distribution which is also, up to the sign, that of the conducting electrons in the normal state.

In technical terms, the electric penetration length is identified directly from the effective potential rather that the full effective action, through the dependence of the effective potential (20) on the chemical potential. Indeed, the chemical potential adds up with the electrostatic potential, and the actual effective action is then function of the electrochemical potential. Through an expansion in the electrostatic potential, one then identifies the electric penetration length. More specifically, given the effective action computed as indicated above through the path integral over the fermionic degrees of freedom, one still needs to add to it the Hamiltonian or energy density of the purely electromagnetic sector, which is treated semi-classically in the effective field theory approach. The latter reads

$$
\begin{aligned}
& \int d^3\vec{r} \left\{ \tfrac{1}{2}\vec{E}^2 + \tfrac{1}{2}\vec{B}^2 - \Phi \left(\vec{\nabla} \cdot \vec{E} - \rho_{\text{tot}} \right) \right\} \\
& = \int d^3\vec{r} \left\{ \tfrac{1}{2} \left(\vec{E} + \vec{\nabla}\Phi \right)^2 + \tfrac{1}{2}\vec{B}^2 - \tfrac{1}{2} \left(\vec{\nabla}\Phi \right)^2 + \Phi\rho_{\text{tot}} \right\},
\end{aligned}
\tag{27}
$$

where ρ_{tot} stands for the total charge density in the conductor, inclusive of the background lattice and valence electron contributions (the "static" charges), to which those of conducting electrons modeled through the above discussion are to be added. Consequently, for what concerns a stationary configuration, by adding this contribution to that following from the effective action one is left with a local functional of the form

$$
\mathcal{L}_{\text{tot}} = f \left| \left(\vec{\nabla} - 2ie\vec{A} \right) \Delta \right|^2 + \Phi\rho_{\text{tot}} - \frac{1}{2}g\Phi^2 - \frac{1}{2} \left(\vec{\nabla}\Phi \right)^2 + \frac{1}{2} \left(\vec{\nabla} \times \vec{A} \right)^2, \tag{28}
$$

where f and g are quite involved expressions determined from the effective action calculation. Note well however that the term in $g\Phi^2$ derives solely from the effective *potential* rather than the full effective action, whereas the term involving f is a contribution from the effective action *per se* which determines the magnetic penetration length. Deriving now the equation of motion for the electrostatic potential,

$$
\vec{\nabla}^2\Phi - g\,\Phi + \rho_{\text{tot}} = 0, \tag{29}
$$

it is quite clear that the electric field penetration length is determined by the coefficient g through

$$
\frac{1}{\lambda_{\text{electric}}^2} = g. \tag{30}
$$

Hence indeed the electric field penetration length is solely determined from the expansion to second order of the effective potential with respect to the chemical potential, since in the presence of an external electrostatic potential the effective potential is function of the electrochemical potential, namely the sum of both the chemical and electrostatic potentials. Finally, the explicit analysis finds that g does receive two types of contributions, one which vanishes for a vanishing order parameter Δ_0, *i.e.*, in the absence of Cooper pairs or the "superconducting electron" contribution, and the second which remains finite even when $\Delta_0 = 0$, namely the "normal electron" contribution, which for all practical purposes leads in effect to the Thomas–Fermi length.

Given this fact, it thus appears that a similar calculation is perfectly feasible also in a nonrelativistic setting, since one only needs to consider the dependence of the effective potential on the chemical potential. Nevertheless, and somewhat suprisingly perhaps, this dependence does not appear to ever have been studied previously, and our result is thus totally new in the literature.

Given that the Thomas–Fermi screening length increases with the depletion of conducting electrons, *i.e.*, by charging positively the conductor, it would appear that possibly one could in effect remove the effect of the "normal" electrons and thus reach a regime in which both the magnetic and electric penetration lengths have comparable values, enabling an experimental confirmation of the effects of electric fields in a set-up of the type used in our experiments. Unfortunately, a detailed analysis of the dependence on the chemical potential μ_0 of both these penetration lengths given our explicit resuls, has established that such a regime is never achieved. Even though both lengths essentially diverge when the conductor is totally depleted of its conducting electrons, their ratio never approaches a value close to unity, rather it essentially retains its value for the neutral conductor. Likewise for what concerns their dependence on temperature, although the magnetic penetration length diverges close to T_c, the electric one does not display any particular behaviour when crossing the critical threshold because of the dominance of the Thomas–Fermi screening length, and one reproduces the correct values and temperature dependence of the latter quantity in the normal state as well.

Finally, our analysis has provided for the first time the temperature dependence of the coherence length of a superconductor such as aluminium. Even though the numerical values obtained for that material coincide with measured ones, our explicit expressions for that quantity lead to an un-

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expected behaviour of that observable when approaching the critical temperature. Indeed, while it remains rather stable at low temperatures, upon approaching T_c, one observes first a slight dip (on the order to 10%) in its value before increasing as expected phenomenologically within the GL framework as T_c is reached. Note that in contradistinction to the magnetic penetration length which is accessible experimentally, there does not appear to exist measurements available in the literature of the temperature dependence of the coherence length of superconductors. This unexpected behaviour of the coherence length requires further corroboration, both experimental and theoretical.

6. Conclusions and Prospects

In this brief contribution, we have described some of the results achieved through a relativistic invariant extension of the well established BCS theory of LTc superconductivity. The motivation for this study is a better understanding, in a relativistic regime at a later dynamical stage, of the effects not only of magnetic fields but also of electric fields on the superconducting state. Following experimental measurements performed on nanoscopic superconductors of which the results were totally unexpected, it was realised that the role of "normal" electrons is also crucial for what concerns such electric field effects. When these are properly included within a microscopic framework, it appears that ordinary screening effects in conductors overwhelm the properties stemming from the superconducting state, an occurrence which does not apply to magnetic field effects for which only the contributions from "superconducting" electrons are relevant. Having identified precisely the origin of the different contributions to the total electric penetration length, it appears that, at least in the instance of that specific observable, a nonrelativistic analysis of the effective potential and its dependence on the chemical potential would have sufficed to reach the same conclusion. In the regime of temperatures and chemical potentials of relevance to ordinary superconductors, indeed the effect of positron-like states is perfectly insignificant. There exist other physical environments though, for which this would no longer be the case, for instance in the astrophysical context.

Our work leaves open a series of issues and even possibilities of detailed study which deserve to be investigated further. For instance, our analysis predicts that under certain circumstances superconductors would acquire locally on their surface nonvanishing charge. Since in recent years such measurements have become possible, a detailed analysis of this issue would

be of interest with the prospect of experimental validation of the model. An unexpected temperature dependence of the coherence length has been identified, which deserves confirmation. The relativistic framework has also led to further possible types of order parameters than simply the s-wave BCS one, including p- and d-wave order parameters. As is well known, High Temperature (HTc) superconductors display properties of mixtures of s-, p and d-wave order parameters, in combinations depending on the material being considered. Would the present framework enable a description of some of these HTc superconductors? Note also that by combining now in the effective four-fermion interaction a superposition of the three types of order parameters, one would obtain a description of systems possessing more than one gap, indeed as also been observed for some HTc materials. There is thus a rich phenomenology of properties to be described through such generalisations of our work. Finally, one may still extend further the choice of four-fermion interaction, by including higher derivative couplings or introducing Lorentz noninvariant couplings. Indeed after all, the thermodynamical description remains tied up with the rest frame of the material being considered, and from that point of view one may still extend the range of possible four-fermion interactions, and see whether some classes of models could account for the observed properties of new classes of superconductors.

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